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Arithmetical properties of p -adic numbers related to numerical systems

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Abstract

In this paper we investigate lower bounds for the numbers of nonzero digits of p -adic algebraic numbers. As a consequence, we introduce criteria for the transcendence of certain power series and we give new examples of transcendental numbers.

1 Introduction

Many mathematicians have studied the transcendence of the values of analytic functions $f(z)$ at algebraic points. In this paper we consider the case where

$$f(z) = \sum_{n=0}^{\infty} s_n z^n, \quad (1.1)$$

where $(s_n)_{n=0,1,\dots}$ is a bounded sequence of nonnegative integers. We denote by $\mathcal{M}(\mathbb{Q}) = \{\infty\} \cup \{p \mid p \text{ is prime}\}$ the set of the places of \mathbb{Q} . Let $v \in \mathcal{M}(\mathbb{Q})$. We denote by \mathbb{C}_v a completion of an algebraic closure of \mathbb{Q}_v . If $v = \infty$, then let $|\cdot|_{\infty}$ be the usual Euclidean norm on \mathbb{C} . In the case where $v = p$ is a prime, $|\cdot|_p$ denotes the p -adic norm on \mathbb{C}_p normalized so that $|p|_p = p^{-1}$.

Let $v \in \mathcal{M}(\mathbb{Q})$ be fixed. Let $(v_m)_{m=0,1,\dots}$ be a sequence of nonnegative integers satisfying

$$v_{m+1} > v_m \quad (1.2)$$

for any sufficiently large m . Then the power series $\sum_{m=0}^{\infty} z^{v_m}$ is rewritten as (1.1), where $s_n \in \{0, 1\}$ for any sufficiently large n . Corvaja and Zannier [9] proved for any algebraic number α with $0 < |\alpha|_v < 1$ that if

$$\liminf_{m \rightarrow \infty} \frac{v_{m+1}}{v_m} > 1, \quad (1.3)$$

then $\sum_{m=0}^{\infty} \alpha^{v_m}$ is transcendental. However, the transcendence of $\sum_{m=0}^{\infty} \alpha^{v_m}$ is generally unknown if (1.3) does not hold. If $v = \infty$ and $\alpha = 1/2$, then Bailey, Borwein, Crandall, and Pomerance [4] improved the criteria by Corvaja and Zannier as follows: If

$$\limsup_{m \rightarrow \infty} \frac{v_m}{m^R} = \infty \quad (1.4)$$

for any positive real number R , then $\sum_{m=0}^{\infty} 2^{-v_m}$ is transcendental. The result above is an application of the results on the binary expansions of algebraic irrational numbers.

In Section 2 we review known results on the β -expansions of algebraic numbers in the case where β is a Pisot or Salem number. In particular, we introduce criteria for the transcendence of power series in [11]. In Section 3 we investigate the digits of p -adic algebraic numbers. Consequently, we deduce criteria for the transcendence of the p -adic numbers, which gives new examples of transcendental numbers.

2 Criteria for transcendence related to β -expansion

In this section we denote the integral and fractional parts of a real number x by $\{x\}$ and $\lfloor x \rfloor$, respectively. Let β be a real number greater than 1 and $T_\beta : [0, 1] \rightarrow [0, 1]$ the β -transformation defined by

$$T_\beta(x) = \{\beta x\}.$$

Then the β -expansion of a real number $\xi \in [0, 1]$ is given by

$$\xi = \sum_{n=1}^{\infty} t_n(\beta; \xi) \beta^{-n},$$

where $t_n(\beta; \xi) = \lfloor \beta T_\beta^{n-1}(\xi) \rfloor$ for any positive integer n .

If $\beta = b$ is a rational integer, then the β -expansion of ξ coincides with the usual base- b expansion of ξ . In particular, the sequence $(s_n(b; \xi))_{n=1,2,\dots}$ is ultimately periodic for any rational number $\xi \in [0, 1]$. Now we recall the definition of Pisot and Salem numbers. Let $\beta > 1$ be an algebraic integer with conjugates $\beta_1 = \beta, \beta_2, \dots, \beta_d$. Then β is a Pisot number if $|\beta_i| < 1$ for any i with $i \geq 2$. In particular, any rational integer $b \geq 2$ is a Pisot number. On the other hand, β is a Salem number if $|\beta_i| \leq 1$ for any i with $2 \leq i \leq d$ and if there exists j with $2 \leq j \leq d$ such that $|\beta_j| = 1$. Schmidt [13] proved that if the β -expansion of each rational number $\xi \in [0, 1]$ is ultimately periodic, then β is a Pisot or Salem number. In the rest of this section, we assume that β is a Pisot or Salem number. Bertrand [5] and Schmidt [13] proved for any Pisot number ξ that any number $\xi \in \mathbb{Q}(\beta) \cap [0, 1]$ has the ultimately periodic β -expansion. However, the periodicity of the β -expansion of rational numbers of ξ is generally not known in the case where β is a Salem number.

In this section we study the numbers of nonzero digits in the β -expansions of algebraic numbers $\xi \leq 1$. Let

$$\nu(\beta, \xi; N) := \text{Card}\{n \in \mathbb{Z} \mid 1 \leq n \leq N, t_n(\beta; \xi) \neq 0\}$$

for any positive integer N . It is generally difficult to give the asymptotic behavior of the function $\nu(\beta, \xi; N)$ for fixed $\beta > 1$ and $\xi \leq 1$. For instance, consider the case where $\beta = b$ is a rational integer. Borel [6] conjectured that any algebraic irrational number is normal. If Borel's conjecture is true, then we have

$$\lim_{N \rightarrow \infty} \frac{\nu(b, \xi; N)}{N} = \frac{b-1}{b} (> 0).$$

However, we know no example of a base $b \geq 2$ and an algebraic irrational number $\xi \leq 1$ such that the inequality

$$\limsup_{N \rightarrow \infty} \frac{\nu(b, \xi; N)}{N} > 0$$

has been proved.

In what follows, we introduce known results on the lower bounds for $\nu(\beta, \xi; N)$. Unless otherwise specified, " $\nu(\beta, \xi; N) \gg f(N)$ for any sufficiently large N " implies that there exist effectively computable positive constants $C(\beta, \xi)$ and $C'(\beta, \xi)$ depending only on β and ξ satisfying

$$\nu(\beta, \xi; N) \geq C(\beta, \xi)f(N)$$

for any integer $N \geq C'(\beta, \xi)$. Bugeaud [8] gave lower bounds for the digit changes in the β -expansions of algebraic numbers β and ξ . In the case where β is a Pisot or Salem number, using his results, we get the following: Let $\xi \in [0, 1]$ be an algebraic number. Suppose that $t_n(\beta; \xi) \neq t_{n+1}(\beta; \xi)$ for infinitely many n 's. Then

$$\nu(\beta, \xi; N) \gg \frac{(\log N)^{3/2}}{(\log \log N)^{1/2}} \quad (2.1)$$

for any sufficiently large N .

In the case where $\beta = 2$, then Bailey, Borwein, Crandall and Pomerance [4] gave better lower bounds as follows: For any algebraic irrational number ξ of degree D , we have

$$\nu(2, \xi; N) \geq C_1(\xi)N^{1/D} \quad (2.2)$$

for any integer $N \geq C_2(\xi)$, where $C_1(\xi)$ is an effective positive constant and C_2 is an ineffective positive constant. $C_1(\xi)$ and $C_2(\xi)$ depend only on ξ .

Consequently, we deduce the criteria for transcendence we mentioned in Section 1. In fact, let $(v_m)_{m=0,1,\dots}$ be a sequence of nonnegative integers satisfying (1.2) for any sufficiently large integer m and (1.4) for any positive real number R . Put $\xi_1 := \sum_{m=0}^{\infty} 2^{-v_m}$. Then we have, for an arbitrary positive real number ε ,

$$\liminf_{N \rightarrow \infty} \frac{\nu(2, \xi_1; N)}{N^\varepsilon} = 0.$$

In particular, (2.2) does not hold for any positive integer D . Hence, ξ_1 is transcendental.

Adamczewski, Faverjon [1] and Bugeaud [8] independently gave effective versions of (2.2) for any integral base as follows: Let $\beta = b \geq 2$ be any integer and $\xi \in [0, 1]$ an irrational number of degree D . Then

$$\nu(b, \xi; N) \gg N^{1/D} \quad (2.3)$$

for any sufficiently large N .

In the case where β is a general Pisot or Salem number, then an analogy of (2.2) and (2.3) is obtained by the following criteria in [11]: Let B be a positive integer and $(s_n)_{n=0,1,\dots}$ a sequence of integers such that

$$0 \leq s_n \leq B$$

for any nonnegative integer n . Put

$$\xi := \sum_{n=0}^{\infty} s_n \beta^{-n} \quad (2.4)$$

and

$$\tilde{\nu}(N) := \text{Card}\{n \in \mathbb{Z} \mid 0 \leq n \leq N, s_n \neq 0\}.$$

Note that (2.4) is not generally the β -expansion of ξ . Suppose that ξ is an algebraic number satisfying $[\mathbb{Q}(\beta, \xi) : \mathbb{Q}(\beta)] = D$, where $[L : K]$ denotes the degree of a field extension K/L . Then there exist effectively computable positive constants $C_3(\beta, \xi, B)$ and $C_4(\beta, \xi, B)$ depending only on β, ξ , and B such that

$$\tilde{\nu}(N) \geq C_3(\beta, \xi, B) \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}} \quad (2.5)$$

for any integer N with $N \geq C_4(\beta, \xi, B)$. In particular, applying the case where (2.4) is the β -expansion of ξ , we have

$$\nu(\beta, \xi; N) \gg \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}}$$

for any sufficiently large N .

Moreover, we obtain criteria for the transcendence of power series as follows: Let again $(v_m)_{m=0,1,\dots}$ be a sequence of nonnegative integers fulfilling (1.2) and (1.4). Then

$$\xi_2 := \sum_{m=0}^{\infty} \beta^{-v_m} \quad (2.6)$$

is transcendental. Note that (2.6) is not also generally the β -expansion of ξ_2 . In the case where $\beta = b$ is a rational integer, then the criteria above were essentially obtained by Bailey, Borwein, Crandall, and Pomerance [4]. The main purpose of this paper is to give an analogy of (2.5) for p -adic algebraic numbers.

We give examples of transcendental numbers. Put

$$w_m := \lfloor m^{\log m} \rfloor = \lfloor \exp((\log m)^2) \rfloor \quad (2.7)$$

for any $m \geq 1$ and

$$x_m := \lfloor m^{\log \log m} \rfloor = \lfloor \exp((\log m)(\log \log m)) \rfloor \quad (2.8)$$

for any $m \geq 3$. Then we have

$$\lim_{m \rightarrow \infty} \frac{w_m}{m^R} = \infty, \quad \lim_{m \rightarrow \infty} \frac{x_m}{m^R} = \infty$$

for any positive real number R . Hence,

$$y_1 := \sum_{m=1}^{\infty} \beta^{-w_m}, \quad y_2 := \sum_{m=3}^{\infty} \beta^{-x_m}$$

are transcendental for any Pisot or Salem number β . Note that $(w_m)_{m=1,2,\dots}$ and $(x_m)_{m=3,4,\dots}$ do not satisfy (1.3) because

$$\lim_{m \rightarrow \infty} \frac{w_{m+1}}{w_m} = 1, \quad \lim_{m \rightarrow \infty} \frac{x_{m+1}}{x_m} = 1.$$

Therefore, the transcendence of y_1 and y_2 is not obtained by the criteria by Corvaja and Zannier in Section 1.

3 Review of p -adic normal numbers

Let b be an integer greater than 1 and $\xi \leq 1$ a real number. Recall that ξ is normal in base- b if the following property holds: Let L be any positive integer and v any word from the alphabet $\{0, 1, \dots, b-1\}$ with length L . Then v appears in the base- b expansion of ξ with average frequency tending to b^{-L} , namely, we have

$$\lim_{N \rightarrow \infty} \frac{\text{Card}\{n \mid 1 \leq n \leq N, v = t_n(b; \xi) \dots t_{n+L-1}(b; \xi)\}}{N} = \frac{1}{b^L}.$$

Ferrero and Washington [10] introduced the notion of joint normality of p -adic integers, where p is a prime number. For any $\xi \in \mathbb{Z}_p$, we denote the p -adic expansion by

$$\xi = \sum_{n=0}^{\infty} t_n^{(p)}(\xi) p^n,$$

where $t_n^{(p)}(\xi) \in \{0, 1, \dots, p-1\}$ for any nonnegative integer n . For simplicity, put

$$\mathbf{t}_{n,M}^{(p)}(\xi) := t_n^{(p)}(\xi) t_{n+1}^{(p)}(\xi) \dots t_{n+M-1}^{(p)}(\xi)$$

for any integers $n \geq 0$ and $M \geq 1$. In what follows, put

$$\mathcal{A} := \{0, 1, \dots, p-1\}.$$

Let $\xi_1, \dots, \xi_r \in \mathbb{Z}_p$. We say that ξ_1, \dots, ξ_r are jointly normal if the following property holds: Let L be any positive integer and v_1, \dots, v_r be any words from the alphabet \mathcal{A} with length L . Then we have

$$\lim_{N \rightarrow \infty} \frac{\text{Card}\{n \mid 1 \leq n \leq N, v_i = \mathbf{t}_{n,L}^{(p)}(\xi_i) \text{ for any } i = 1, \dots, r\}}{N} = \frac{1}{p^{rL}}.$$

In the case of $r = 1$, if ξ is jointly normal, then we call ξ normal. It is remarkable that the notion of joint normality is applicable to Iwasawa theory. In fact, Ferrero and Washington [10] used certain jointly normal numbers to verify that the Iwasawa invariant $\mu_p(k)$ vanishes for any abelian number field k (see also [15]).

Moreover, Anglès [3] proposed a problem on the transcendence of the Iwasawa power series modulo certain uniformization of $\mathbb{Z}_p[\theta]$ over $\overline{\mathbb{F}_p}(T)$, where $\overline{\mathbb{F}_p}$ is an algebraic closure of the finite field with p elements. Sun [14] proved the transcendence on this problem under

the assumption that Borel's conjecture on the normality of p -adic algebraic irrational number $\xi \in \mathbb{Z}_p$ holds.

Borel's conjecture of p -adic version implies for each prime number p that any algebraic irrational number $\xi \in \mathbb{Z}_p$ is normal. However, this conjecture has not also been proved. For any $\xi \in \mathbb{Z}_p$, we write the complexity function of the sequence $(t_n^{(p)}(\xi))_{n=0,1,\dots}$ by

$$P(\xi; N) := \text{Card}\{t_{n,N}^{(p)}(\xi) \mid n \in \mathbb{Z}, n \geq 0\} \quad (N = 1, 2, \dots).$$

If ξ is normal, then we have $P(\xi; N) = p^N$ for any positive integer N . Adamczewski and Bugeaud [2] verified for any algebraic irrational number $\xi \in \mathbb{Z}_p$ that

$$\lim_{N \rightarrow \infty} \frac{P(\xi; N)}{N} = \infty. \quad (3.1)$$

In the same paper, they also investigated certain criteria for the transcendence of p -adic integers. For any finite word $W = w_1 \dots w_l$ and any positive real number x , set

$$|W| := l$$

and

$$W^x := \underbrace{WW \dots W}_{[x]} W',$$

where W' is the prefix of W with length $[\{x\}|W|]$. We say for any $\xi \in \mathbb{Z}_p$ that ξ satisfies Condition A if $(t_n^{(p)}(\xi))_{n=0,1,\dots}$ is not ultimately periodic and if there exists a real number $w > 1$ satisfying the following: there exist two sequences of finite words $(U_n)_{n=1,2,\dots}$ and $(V_n)_{n=1,2,\dots}$ from the alphabet \mathcal{A} such that

1. $U_n V_n^w$ is a prefix of the sequence $(t_n^{(p)}(\xi))_{n=0,1,\dots}$;
2. $(|U_n|/|V_n|)_{n=1,2,\dots}$ is bounded;
3. $(|V_n|)_{n=1,2,\dots}$ is strictly increasing.

Adamczewski and Bugeaud [2] proved for any $\xi \in \mathbb{Z}_p$ that if ξ satisfies Condition A, then ξ is transcendental. Finally, we note that (3.1) and the above criteria for transcendence are analogies of β -expansions of real numbers by a Pisot or Salem number β in [2].

4 Criteria for transcendence related to p -adic expansion

In this section, p and d denote a fixed prime number and a fixed positive integer, respectively. Put $L := \mathbb{Q}_p(p^{1/d})$. Then the extension L/\mathbb{Q}_p is totally ramified of degree d and the ring of integers of L is $\mathcal{O} = \mathbb{Z}_p(p^{1/d})$. For any $\xi \in \mathcal{O}$, there exist a unique expansion

$$\xi = \sum_{n=0}^{\infty} t_n^{(p)}(\xi) p^{n/d},$$

where $t_n^{(p)}(\xi) \in \mathcal{A} = \{0, 1, \dots, p-1\}$ for any $n \geq 0$. Set

$$\nu^{(p)}(\xi; N) := \text{Card}\{n \in \mathbb{Z} \mid 0 \leq n \leq N, t_n^{(p)}(\xi) \neq 0\}.$$

We introduce an analogy of the lower bounds for $\nu(\beta, \xi; N)$ in Section 2 as follows:

THEOREM 4.1. *Let $\xi \in \mathcal{O}$ be an algebraic number of degree D . Assume that $t_n^{(p)}(\xi) \neq 0$ for infinitely many n 's. Then there exist effectively computable positive constants $C_5(p, d, \xi)$ and $C_6(p, d, \xi)$ depending only on p, d and ξ such that*

$$\nu^{(p)}(\xi; N) \geq C_5(p, d, \xi) N^{1/D}$$

for any integer N with $N \geq C_6(p, d, \xi)$.

Applying Theorem 4.1, we obtain criteria for transcendence as follows: Let $\xi \in \mathcal{O}$. Suppose that $t_n^{(p)}(\xi) \neq 0$ for infinitely many n and that

$$\liminf_{N \rightarrow \infty} \frac{\nu^{(p)}(\xi; N)}{N^R} = 0$$

for any positive real number R . Then ξ is transcendental.

We give new examples of transcendental numbers in \mathcal{O} . Recall that $(w_m)_{m=1,2,\dots}$ and $(x_m)_{m=3,4,\dots}$ are defined by (2.7) and (2.8), respectively. Then

$$\sum_{m=1}^{\infty} p^{w_m/d}, \sum_{m=3}^{\infty} p^{x_m/d}$$

are transcendental.

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